An Introduction to Complex Analysis

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Resource: Complex Variables and Applications

Introduction

Complex numbers can be defined as ordered pairs (x, y) with $x, y \in \mathbb{R}$. This point is a point on the *complex plane*. The complex plane is constructed by the real x-axis and the imaginary y-axis. Therefore (x, 0) is the real x-axis and (0, y) is the imaginary y-axis and $\mathbb{R} \subset \mathbb{C}$.

It is common to denote a complex number (x, y) by z. If z = (x, y) then Re(z) = x and Im(z) = y [read: real part of z = x and imaginary part of z = y]. Complex numbers are often written as $z = x + yi$.

(1.1) $z_1 = z_2 \; iff \; Re(z_1) = Re(z_2) and \; Im(z_1) = Im(z_2)$

Basic Operations

(1.2) $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

(1.3) $z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$

(1.4) $i^2 = -1$

Commutative Laws

(1.5) $z_1 + z_2 = z_2 + z_1 \; and \; z_1 z_2 = z_2 z_1$

Associative Laws

(1.6) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \; and \; (z_1 z_2) z_3 = z_1 (z_2 z_3)$

Identities

(1.7) $z + 0 = z \; and \; z \cdot 1 = z$

Additive Inverse

(1.7) $-z = (-x, -y), \; z + (-z) = 0$

Subtraction

(1.8) $z_1 - z_2 = z_1 + (-z_2)$

Multiplicative Inverse

(1.9) $zz^{-1} = 1,\ z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)\ (z \neq 0)$

(1.10) $\frac{z_1}{z_2} = z_1 z_2^{-1}\ (z_2 \neq 0)$

(1.11) $\frac{z_1}{z_2} = \frac{(x_1+y_1 i)(x_2-y_2 i)}{(x_2+y_2 i)(x_2-y_2 i)}$

Distance and Conjugates

The *modulus* (absolute value) represents the distance from z to the origin. It is defined by

$$(2.1)\quad |z| = \sqrt{x^2 + y^2}$$

Note: There can be no comparisons in the complex plane of $z_1 < z_2$ only $|z_1| < |z_2|$, which states that z_1 is closer to the origin than z_2.

(2.2) $|z|^2 = [Re(z)]^2 + [Im(z)]^2$

The *complex conjugate* is defined and denoted as

$$(2.3)\quad \bar{z} = x - yi$$

Basic Conjugate Relationships

(2.3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2.4) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(2.5) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(2.6) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}\ (z^2 \neq 0)$

(2.7) $z\bar{z} = |z|^2$

Proof (2.7)

Let $z = x + yi$ therefore $\bar{z} = x - yi$.

$z\bar{z} = (x + yi)(x - yi) = x^2 - xyi + xyi - yi^2 = x^2 + y^2$

$|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$

$\therefore\ z\bar{z} = |z|^2$

Prove: z is real iff $\bar{z} = z$

1. Let $z \in \mathbb{R}$
2. $z = x + 0i = x$
3. $\bar{z} = x - 0i = x.$
4. Let $z = x + yi\ and\ \bar{z} = z$.
5. $x + yi = x - yi$
6. $y = -y$
7. y = 0
8. $z \in \mathbb{R}$

QED

Polar Form

Often you will find that working with complex numbers in polar form is much easier or more natural to solve specific types of problems. Let r and θ be polar coordinates of the point (x, y) that correspond to a nonzero complex number $z = x + yi$. Since $x = r\cos\theta\ \ and\ y = r\sin\theta$ z can be written in polar form as

$$(3.1) \quad z = r(\cos\theta + i\sin\theta)$$

In the complex number system r cannot be negative and is the length of the radius vector of $z, |z|$. The real number θ represent the radian angle that z makes with the positive real axis. Because there are an infinite amount of possible angles each angle is an **argument** of z, denoted arg z, and one specific angle is considered to be the **principal value** of arg z, denoted Arg z. Arg z is the unique value Θ, such that $-\pi < \Theta \leq \pi$.

$$(3.2) \quad \text{arg z = Arg z} + 2n\pi\ (n = 0, \pm 1, \pm 2, \dots)$$

Exponential Form

Euler's formula states that a number can be written as

$$(3.3) \quad e^{i\theta} = \cos\theta + i\sin\theta$$

Further written into exponential form as

$$(3.4) \quad z = re^{i\theta}$$

Example 1

Write $z = -1 - i$ in polar form.

$$\text{Arg z} = -\frac{3}{4}\pi$$

$$r = |z| = \sqrt{-1^2 + -1^2} = \sqrt{2}$$

$$z = \sqrt{2}e^{-\frac{3}{4}i\theta}$$

Binomial Formula for Complex Numbers

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} \mathrm{z}_1^{\mathrm{n-k}} \mathrm{z}_2^{\mathrm{k}} \ (\mathrm{n} = 1,2, \ldots)$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \ (k = 0,1,2, \ldots, n)$$